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Aggregation Operations on Multiple Sets

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Abstracts— As a new approach towards the vagueness, multiple sets are introduced. Then it is noticed that multiple sets are generalization of fuzzy sets, fuzzy multisets, multi fuzzy sets and multisets. Standard set operations subset, complement, union, intersection are defined on multiple sets. Aggregation operations on multiple sets are defined and their properties are explored.

Index Terms— Multiple sets, fuzzy sets, multisets, fuzzy multisets, multi fuzzy sets, aggregation operations, averaging operations.

1 INTRODUCTION

Vagueness, imprecision, inexactness, ambiguity and uncertainty are part of our daily life. A lot of new mathematical constructions and theories treating these concepts have been developed. Fuzzy sets[8], multisets[2], fuzzy multisets[7], multi fuzzy sets[6], L-fuzzy sets[3], intuitionistic fuzzy sets[1], rough sets[5] are some of them. As a part of these theories, multiple sets are introduced. That is, a multiple set is a new approach to represent vague data. Also, fundamental operations on multiple sets are defined.

Aggregation is the state of being so collected. As the term indicates, aggregation operations group values from multiple documents together, and can perform a variety of operations on the grouped data to return a single result. That is, aggregations operations process data records and return computed results. In this paper, aggregation operations on multiple sets are defined, which can be considered as the generalization of the aggregation operations on fuzzy sets. Averaging operations, a special type of aggregation operations which takes values between standard minimum operation and maximum operation, are also defined.

2 PRELIMINARIES

The basic concepts of fuzzy sets, multisets, fuzzy multisets, multi fuzzy sets, aggregation operations on fuzzy sets are introduced below to facilitate future discussions.

Definition 2.1.[8] Let X be a given universal set, which is always a crisp set. A **fuzzy set** A on X is characterized by a function $A : X \to [0,1]$ called fuzzy membership function, which assigns to each object a grade of membership ranging between zero and one. A fuzzy set A is defined as

$$A = \{ (x, A(x)) ;\ x \in X \}$$

where $A(x)$ is the fuzzy membership value of x in X.

Definition 2.2.[2] Let X be a non empty set, called universe. A **multiset** M drawn from X is represented by a count function $C_M : X \to \mathbb{N} \cup \{0\}$ where $\mathbb{N}$ is the set of all positive integers. For each $x \in X$, $C_M(x)$, indicates the number of occurrences of the element x in M. Then a multiset M can be expressed as $\left\{ {}^{C_M}/_{x} ;\ x \in X \right\}$.

Definition 2.3.[7] For $x \in X$, the membership sequence of x is defined as a non increasing sequence of membership values of x and it is denoted by $\{\mu_A^1(x), \mu_A^2(x) \ldots\ldots\ldots \mu_A^k(x)\}$ such that $\mu_A^1(x) \geq \mu_A^2(x) \geq \cdots . \geq \mu_A^k(x)$ where μ_A is a membership function and μ_A^j , $j = 1, 2 \ldots\ldots k$ are values(same or different) of membership function μ_A. A **fuzzy multiset** is a collection of all x together with its membership sequence.

Defintion 2.4.[6] Let X be a non empty set and let $\{L_i, i \in \mathbb{N}\}$ be a family of complete lattices where $\mathbb{N}$ is the set of positive integers. A **multi fuzzy set** A in X is a set of ordered sequences $\{(x, \mu_1(x), \mu_2(x), \ldots\ldots); x \in X\}$, where $\mu^i \in L_i^X$ for $i \in \mathbb{N}$. The function $\mu_A = (\mu_1, \mu_2, \ldots\ldots)$ is called a multi membership function of multi fuzzy set A.

Defintion 2.5.[4] An aggregation operation on m fuzzy sets $(m \geq 2)$ defined by a function $h : [0,1]^m \to [0,1]$ satisfying the axioms:

Axiom h1: Bounadary conditions

$$h(\langle 0, \ldots\ldots 0 \rangle) = 0$$
$$h(\langle 1, \ldots\ldots 1 \rangle) = 1$$

Aiom h2: For any pair $\langle a_1, a_2, \ldots\ldots, a_m \rangle$ and $\langle b_1, b_2, \ldots\ldots, b_m \rangle$ in $[0,1]^m$ and if $a_i \leq b_i$ for all $i \in N_m$, then

$$h(\langle a_1, a_2, \ldots\ldots. a_m \rangle) \leq h(\langle b_1, b_2, \ldots\ldots b_m \rangle)$$

that is, h is monotonically increasing in all its arguments.

Axiom h3: h is a continuous function.

Besides these essential requirements, aggregation operations on fuzzy sets are usually expected to satisfy two additional axiomatic requirements;

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Axiom h4: h is a symmetric function in all its arguments; that is,

$$h(\langle a_1, a_2, \ldots \ldots a_m \rangle) = h(\langle a_{p(1)}, a_{p(2)}, \ldots \ldots a_{p(m)} \rangle)$$

for any permutation p on N_m.

Axiom h5: h is an idempotent function; that is,

$$h(\langle a, a, \ldots \ldots a \rangle) = a \text{ for all } a \in [0,1]$$

3 MULTIPLE SETS

In multiple sets, multiple occurrences of elements are permitted in which each occurrence has a finite number of same or different membership values. That is, in multiple set theory, a multiple set of order (n,k) gives nk membership grades to each element x in the universal set X.

Defintion 3.1. Let X be a non empty set called universe. A multiple set A drawn from X is an object of the form $\{(x, A(x)) ; x \in X\}$, where for each $x \in X$, its membership value is an $n \times k_x$ matrix

$$A(x) = \begin{bmatrix} A_1^1(x) & A_1^2(x) & \ldots & A_1^{k_x}(x) \\ A_2^1(x) & A_2^2(x) & \ldots & A_2^{k_x}(x) \\ \ldots & \ldots & \ldots & \ldots \\ A_n^1(x) & A_n^2(x) & \ldots & A_n^{k_x}(x) \end{bmatrix}$$

where A_i, $i = 1,2,\ldots..n$ are membership functions. For each $i = 1,2,\ldots\ldots n$, $A_i^{\,j}(x)$, $j = 1,2,\ldots..k$ are membership values of the membership function A_i for the element $x \in X$. Then, by taking $k = \sup\{k_x; x \in X\}$, we can make $A(x)$ as $n \times k$ matrix by adding sufficient number of columns of zeros. That is, for each $x \in X$

$$A(x) = \begin{bmatrix} A_1^1(x) & A_1^2(x) & \ldots & A_1^{k}(x) \\ A_2^1(x) & A_2^2(x) & \ldots & A_2^{k}(x) \\ \ldots & \ldots & \ldots & \ldots \\ A_n^1(x) & A_n^2(x) & \ldots & A_n^{k}(x) \end{bmatrix}$$

is the membership matrix of order $n \times k$. Then A is called multiple set of order (n,k).

The univeral set X and empty set Φ can be consider as multiple sets with the membership matrix in which all entries are one and zero respectively.

Let A and B be two multiple sets of order (n_1, k_2) and (n_2, k_2) respectively. Take $n = \max(n_1, n_2)$ and $k = \max(k_1, k_2)$.

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Then for each $x \in X$, both $A(x)$ and $B(x)$ can be viewed as $n \times k$ matrices by adding sufficient number of rows and columns of zeroes, if necessary. In this way, we can consider both A and B as multiple sets of order (n,k).

Note that a multiple set A can be viewed as a function $A : X \to M$, where $M = M_{n\times k}([0,1])$ is the set of all matrices of order $n \times k$ with entries from $[0,1]$, which maps each $x \in X$ to its membership matrix $A(x)$. Denote the set of all multiple sets drawn from X by $M\,S(X)$ and set of all multiple sets of order (n,k) drawn from X by $MS_{(n,k)}(X)$.

Remark 3.2. A multiple set A of order $n \times k$ can be viewed as a generalization of fuzzy sets, multi fuzzy sets, fuzzy multisets and multisets.

1. If $k = 1$ and $n = 1$, then A is a fuzzy set. This is a spe cial case of a crisp set.
2. If $n = 1$, then A is a fuzzy multiset.
3. If $k = 1$, then A is a multi fuzzy set (finite case).
4. If $A_j(x) = 0$ or 1 for every $i = 1,2,\ldots,n$ and $j = 1,2,\ldots,k$, then A is a multiset.

6 OPERATIONS ON MULTIPLE SETS

In this section, we introduce the standard multiple set operations that are the generalization of the standard operations complement, intersection and union on crisp sets.

Definition 4.1. Let X be a universal set and $A, B \in MS_{(n,k)}(X)$ and

$$A(x) = \begin{bmatrix} A_1^1(x) & A_1^2(x) & \ldots & A_1^{k}(x) \\ A_2^1(x) & A_2^2(x) & \ldots & A_2^{k}(x) \\ \ldots & \ldots & \ldots & \ldots \\ A_n^1(x) & A_n^2(x) & \ldots & A_n^{k}(x) \end{bmatrix}$$

$$B(x) = \begin{bmatrix} B_1^1(x) & B_1^2(x) & \ldots & B_1^{k}(x) \\ B_2^1(x) & B_2^2(x) & \ldots & B_2^{k}(x) \\ \ldots & \ldots & \ldots & \ldots \\ B_n^1(x) & B_n^2(x) & \ldots & B_n^{k}(x) \end{bmatrix}$$

be the membership matrices for x in A and B respectively.

1. Subset: $A \subseteq B$ if and only if $A_i^{\,j}(x) \le B_i^{\,j}(x)$ for every $x \in X$, $i = 1,2,\ldots,n$ and $j = 1,2\ldots.k$.

2. Equality: $A = B$ if and only if $A \subseteq B$ and $B \subseteq A$.

That is, if and only if $A_i^j(x) = B_i^j(x)$ for every $x \in X$, $i = 1,2,\ldots,n$ and $j = 1,2,\ldots,k$.

3. Standard Union: The union of A and B, denoted by $C = A \cup B$ is a multiple set given by the membership matrix $C(x) = \left(C_i^j(x)\right)$ for every $x \in X$ where $C_i^j(x) = \max\{A_i^j(x), B_i^j(x)\}$ every $i = 1,2,\ldots,n$ and $j = 1,2,\ldots,k$

4. Standard Intersection: The intersection of A and B, denoted by $C = A \cap B$ is a multiple set given by the membership matrix $C(x) = (C_i^j(x))$ for every $x \in X$ where $C_i^j(x) = \min\{A_i^j(x), B_i^j(x)\}$ every $i = 1,2,\ldots,n$ and $j = 1,2,\ldots,k$

5. Standard Complement: The standard complement denoted by $\bar{A}$ is a multiple set given by the membership matrix

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$\bar{A}(x) = \left(\bar{A}_i^j(x)\right)$ for every $x \in X$ where $\bar{A}_i^j(x) = 1 - A_i^j(x)$ every $i = 1,2,\ldots,n$ and $j = 1,2,\ldots,k$.

6 AGGREGATION OPERATIONS

Let $\mathbb{M} = \mathbb{M}_{n\times k}[0,1]$ denotes the set of all matrices of order $n \times k$ with entries from [0,1] and $\mathbb{M}^m$ denotes the m-tuple of matrices in $\mathbb{M}$.

Defintion 5.1. An aggregation operation on m multiple sets $(m \geq 2)$ is defined by a function

$$h : \mathbb{M}^m \rightarrow \mathbb{M}$$

An element $\langle A_1, A_2, \ldots. A_m\rangle \in \mathbb{M}^m$ is mapped to $A \in \mathbb{M}$ in such a way that

$$a_{ij} = h_{ij}(\langle(a_1)_{ij}, (a_2)_{ij}, \ldots.(a_m)_{ij}\rangle)$$

for every $i = 1,2,\ldots,n$ and $j = 1,2,\ldots,k$ and h_{ij} are fuzzy aggregation operators. Clearly A, $A_1, \ldots\ldots A_m$ are matrices given by $A = (a_{ij})$, $A_1 = ((a_1)_{ij}), \ldots\ldots A_m = ((a_m)_{ij})$ We represent multiple aggregation function h as $h = (h_{ij})$.

When applied to multiple sets $A_1, \ldots\ldots A_m$ on X, the function h produces an aggregate multiple set A by operating on membership matrices of these sets for each $x \in X$ Thus

$$A(x) = h$$

for each $x \in X$.

An aggregation operation $h = (h_{ij})$ on multiple sets is said to be symmetric, if each h_{ij} is symmetric and is idempotent, if each h_{ij} is idempotent.

6 AVERAGING OPERATIONS

Any idempotent aggregation operation $h = (h_{ij})$ satisfies the inequalities

$$\min(\langle A_1, A_2, \ldots. A_m\rangle) \leq h(\langle A_1, A_2, \ldots. A_m\rangle)$$

$$\leq max(\langle A_1, A_2, \ldots. A_m\rangle) \qquad (6.1)$$

for all m-tuples $\langle A_1, A_2, \ldots. A_m\rangle \in \mathbb{M}^m$ Also any aggregation operation h that satisfies (6.1) is idempotent. Thus, all aggregation operations between standard multiple intersection and multiple union are idempotent. The functions h that satisfies (6.1) are the only aggregation operations that are idempotent. These aggregation operations are usually called **averaging operations**.

Example 6.1. The aggregation function $h = (h_{ij})$ given by

$$h_{ij}(a_1, a_2, \ldots\ldots. a_m) = \left(\frac{a_1^{\alpha_{ij}} + a_2^{\alpha_{ij}} + \cdots\ldots + a_m^{\alpha_{ij}}}{m}\right)^{\frac{1}{\alpha_{ij}}} \qquad (6.2)$$

where $\alpha_{ij} \in \mathbb{R}(\alpha_{ij} \neq 0)$ and $\alpha_i \neq 0$ for all $i \in \mathbb{N}_m$, when $\alpha_{ij} < 0$ is an averaging operation.

When each $\alpha_{ij} \rightarrow 0$ for all $i = 1,2,\ldots\ldots n$ and $j = 1,2,\ldots..k$, the function $h = (h_{ij})$ given by (6.2) coincides with the geometric mean operation

$$h_{ij} = (a_1 a_2,,, a_m)^{\frac{1}{m}}$$

When each $\alpha_{ij} \rightarrow -\propto$ for all $i = 1,2,\ldots\ldots n$ and $j = 1,2,\ldots..k$,, the function $h = (h_{ij})$ given by (6.2) coincides with the standard minimum operation and when each $\alpha_{ij} \rightarrow \propto$ it coincides with the standard maximum operation. Also, when $\alpha_{ij} = 1$, $i = 1,2,\ldots\ldots n$ and $j = 1,2,\ldots..k$,, the function $h = (h_{ij})$ given by (6.2) coincides with the arithmetic mean operation

$$h_{ij} = \frac{a_1 + a_2 +,,, + a_m}{m}$$

and also, when $\alpha_{ij} = -1$, for every $i = 1,2,\ldots\ldots n$ and $j = 1,2,\ldots..k$, the function $h = (h_{ij})$ given by (6.2) coincides with the harmonic mean operation

$$h_{ij} = \frac{m}{\frac{1}{a_1} + \frac{1}{a_2} + \cdots\ldots. + \frac{1}{a_m}}$$

Illustration 1. The performance of a student x was evaluated in three alternative months of a semester, under the criteria of academic skills and extra curricular activities. The performance of the student x can be represented by a multiple set of order (2,3) as follows: $A_1^j(x)$ $j = 1,2,3$ are membership values correspoding to three alternative months of a semester, for the student x, given to academic skills and $A_2^j(x)$ $j = 1,2,3$ are membership values correspoding to three alternative months of a semester, for the student x, given to extra curricular activities. Suppose A_1, A_2, A_3, A_4 denotes the four multiple sets, which represents the performance of the student in four semester of a course. Appropriate aggregation operation would produce a meaningful expression, in terms of a single multiple set of overall performance of the student x at the end of the fourth semester. For example, consider

$$A_1(x) = \begin{bmatrix} 0.8 & 0.3 & 0.5 \\ 0.7 & 0.5 & 0.5 \end{bmatrix}$$

$$A_2(x) = \begin{bmatrix} 0.8 & 0.9 & 0.3 \\ 0.5 & 0.6 & 0.7 \end{bmatrix}$$

$$A_3(x) = \begin{bmatrix} 0.7 & 0.5 & 0.9 \\ 0.8 & 0.5 & 0.5 \end{bmatrix}$$

$$A_4(x) = \begin{bmatrix} 0.8 & 0.2 & 0.3 \\ 0.8 & 0.7 & 0.9 \end{bmatrix}$$

which represents the performance of the student x at the end of each semester. Consider the arithmetic mean operation as the aggregation operation and it yields single multiple set as,

$$A(x) = \begin{bmatrix} 0.775 & 0.475 & 0.5 \\ 0.7 & 0.575 & 0.65 \end{bmatrix}$$

which represents the overall performance of the student x at the end of the fourth semester.

7 Conclusion

In this paper, we have introduced multiple sets, a new mathematical approach to model vagueness and multiplicity. Multiple set is an extension of fuzzy sets, multisets, fuzzy multisets and multi fuzzy sets. We have introduced aggregation operation on multiple sets as an extension of aggregation operation on fuzzy sets.

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